A Generalized Boundary Element Method for Groundwater Flow in Orthotropic Heterogeneous Media

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ABSTRACT

A boundary element method for the solution of two- and three-dimensional problems of groundwater flow in orthotropic heterogeneous media is developed. A generalized fundamental solution is derived for the governing equation of Darcy flow using a singular, non-symmetric generalized forcing function with special sampling properties. The fundamental solutions are constructed as locally radially symmetric responses to this forcing function. The forcing function and the fundamental solution are both functions of the heterogeneous material property (i.e. hydraulic conductivity) of the media. A boundary integral equation is formulated and implemented in a numerical scheme for the solution of well-posed boundary value problems. Several numerical examples are presented with comparisons to analytical results which illustrate the accuracy of the method. Other numerical applications are provided to show the potential for this general algorithm in solving problems of flow in heterogeneous soil media.

KEYWORDS

porous media, boundary element methods, Darcy flow, non-homogeneous flow properties, orthotropic flow.

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INTRODUCTION

The boundary element method (BEM) has been applied with a great deal of success to problems of groundwater flow in porous media with constant or piecewise continuous permeability (Liggett and Liu, 1983). Such problems in generally non-homogeneous media have traditionally been solved using other numerical techniques such as the finite difference method (FDM) or the finite element method (FEM). The BEM has a striking advantage over these methods for many problems of engineering because it is a Green's function-based integral equation formulation for the solution of the governing partial differential equation. The resulting boundary integral equation is solved using numerical procedures, and only the boundary of the domain needs to be discretized. This is true for linear and nonlinear steady state problems of groundwater flow and in problems of linear elastostatics.

For flow through non-homogeneous media, a domain integral appears in the formulation arising from the spatial variation of the permeability of the medium. In the past, these problems have been solved using either by the discretization of the entire domain to evaluate the domain integral (Lennon, 1984) or by sectioning the domain into multiple zones with constant properties. The multi-zone technique is discussed in the textbook by Gipson (1989) on the solution of the Poisson's equation. In this approach, the boundary integral equations are written for each zone and subsequently the system of equations is coupled by enforcing the continuity of the piezometric head and the fluid flux at the interface between the zones. The resulting block matrix can be solved using an efficient solver.

Recent literature on non-homogeneous flow in porous media has focused on several approaches such as a functionally specified variable permeability (Cheng 1984, 1987), multiple layers with leaky aquifers (Cheng and Morohunfola, 1993) and stochastic flows (Cheng and Lafe, 1991; Lafe and Cheng, 1995). Cheng (1984) has presented an exact Green's function-based method for Darcy's flow with variable permeability using a formulation that requires the determination of Green's functions for each type of permeability variation.

An analogous development of Green's function for some heat conduction problems in heterogeneous media has also been presented by Clements and Larsson (1993). This method eliminates the need for domain discretization and is very attractive, however, the method is restricted to certain forms for the spatial variations of the conducitivity. Recently, Shaw (1993) and Shaw and Gipson (1996) have published a new free space Green's function for potential problems in two and three dimensional heterogeneous media. However, the spatial variation of the properties was linear in one dimension only. Another method applied to the solution of such problems is the perturbation method (Lafe and Cheng, 1987) which is based upon small differences between the solutions for homogeneous and heterogeneous media. This condition may not be satisfied in all cases and as such, perturbation methods are not sufficiently robust.

The dual reciprocity BEM (DRBEM) (Partridge *et al.*, 1992) has provided a mechanism for solving non-homogeneous problems using BEM without resorting to domain discretization. The non-homogeneous part of the governing equation which gives rise to a domain integral in the boundary integral equation is interpolated by a set of basis functions. For groundwater flow, the first implementation of DRBEM for heterogeneous media was done by El Harrouni *et al*. (1995) using polynomial based global interpolation functions developed by Cheng *et al* (1994). Various other types of basis functions may also be used including trigonometric and hyperbolic functions (Cheng *et al*., 1994) and wavelet basis functions (Lafe and Cheng, 1995).

From the above discussion, it is evident that there is no general way to address this very important class of problems using BEM. Recently, Kassab and Divo (1996) have developed a generalized fundamental solution for the solution of heat conduction problems in isotropic, non-homogeneous media using BEM. The aim of the present paper is to extend this formulation to problems of groundwater flow in porous media in which the hydraulic conductivity variations are orthotropic and heterogeneous. We develop a formulation which uses a singular non-symmetric forcing function D (Kassab and Divo, 1996) to derive a boundary-only integral equation along with the generalized fundamental solution E. The sampling properties of the forcing function D are critical to the solution of the integral equation. Both D and E are defined in terms of the hydraulic conductivity of the medium. For the limiting case of constant permeability, the familiar Green's free-space solution for the potential flow problems is recovered. Both two- and three-dimensional fundamental solutions are formulated. The boundary integral equation is discretized using quadratic isoparametric elements, and numerical examples are presented to validate the algorithm. Several examples are presented which validate the formulation for orthotropic heterogeneous flow in both two and three dimensions. The seepage flow beneath a dam and an actual field problem of a heterogeneous aquifer with irregular boundaries are studied. Results obtained from the newly formulated BEM analysis are found to be in excellent agreement with available (analytical or finite element) results.

GOVERNING EQUATIONS FOR FLOW IN A CONFINED AQUIFER

The governing equation for steady state groundwater flow in an orthotropic non-homogeneous confined aquifer is given by Darcy's equation:

$$\frac{\partial}{\partial x_i}[k_{ij}(x)\frac{\partial h}{\partial x_j}(x)] = 0 \tag{1}$$

where $k_{ij}(x)$ is the spatially varying orthotropic hydraulic conductivity of the aquifer such that $k_{ij}(x) = \begin{bmatrix} k_{xx}(x) & 0 \\ 0 & k_{yy}(x) \end{bmatrix}$ and $h(x)$ is the piezometric head. The spatial variable x defines the coordinate system of choice and depends upon the dimension of the problem. Thus, x represents (x_1, x_2) in two dimensions and (x_1, x_2, x_3) in three dimensions. The boundary conditions for this equation can be one or more of these types:

(a) Dirichlet conditions : $h = \overline{h}$ on Γ_1

(b) Neumann conditions: $\frac{\partial h}{\partial n} = \overline{h_n}$ on Γ_2

(c) Robin conditions: $\frac{\partial h}{\partial n} = ah + b$ on Γ_3

where n is the unit outward normal to the boundary Γ, h and $\overline{h_n}$ are prescribed values of the head and its normal gradient on different parts of the surface, and a and b are constants. The above equation is partial differential equation with variable coefficients and will be converted to an integral equation in the next section.

BOUNDARY INTEGRAL EQUATION FOR FLOW IN NON-HOMOGENEOUS ORTHOTROPIC MEDIA

Using conventional integral equation methodology, the governing equation (1) is multiplied by a function $G(x,\xi)$, where ξ is the source point location, and the product is integrated over the domain Ω of the problem as follows:

$$\int_{\Omega} \{G(x,\xi)\frac{\partial}{\partial x_i}[k_{ij}(x)\frac{\partial h(x)}{\partial x_j}]\}d\Omega = 0 \tag{2}$$

Using Green's first identity twice, the following integral equation may be derived

$$\begin{aligned} &\oint_{\Gamma} [G(x,\xi)\,k_{ij}(x)\frac{\partial h(x)}{\partial x_j}n_i - h(x)\,k_{ij}(x)\frac{\partial G}{\partial x_j}(x,\xi)\,n_i]\,d\Gamma + \\ &\int_{\Omega} [h(x)\frac{\partial}{\partial x_i}\{k_{ij}(x)\frac{\partial G}{\partial x_j}(x,\xi)\}]\,d\Omega = 0 \end{aligned} \tag{3}$$

The boundary of the domain is denoted by Γ which has a dimension equal to one less than the dimension of the domain Ω. Next, a solution to the adjoint equation, perturbed by a Dirac Delta function, δ, is introduced as follows

$$\frac{\partial}{\partial x_i}[k_{ij}(x)\frac{\partial G}{\partial x_j}(x,\xi)] = -\delta(x-\xi) \qquad (4)$$

If it possible to obtain an analytical solution to the above equation subject to an appropriate set of boundary conditions, then the function $G(x,\xi)$ is called the Green's function for the problem and the problem is solved exactly. The sampling property of the Dirac Delta function is utilized to convert the domain integral in equation (3) to a point function. Such an exact solution is not always possible usually due to the geometry of the problem. Thus, in conventional BEM, boundary conditions are not imposed and the adjoint equation is solved in infinite domain yielding the Green's free space solution. A boundary integral equation is then derived and subsequently discretized to solve for the unknown flux or head at the boundary.

In case of problems where the permeability does not vary in space, the Green's free space solution for the Darcy equation can be retreived from the Green's function of the Laplace equation upon a subsequent scaling. However, no such general fundamental solution exists for groundwater flow problem in heterogeneous media. This is due to the following reasons: in homogeneous media, the Dirac Delta function is symmetric about the source point ξ, and the adjoint equation is not a variable coefficient partial differential equation, while in the heterogeneous case, the Green's function and the free-space solution is non-symmetric and is more difficult, if not impossible, to derive. Kassab and Divo (1996) proposed a new approach to solve the problem of arbitrary variations of thermal conductivity in heat conduction. The same approach is utilized in this paper to address the groundwater flow problem with spatially varying hydraulic conductivity.

A generalized forcing function $D(x,\xi)$ is introduced (Hoskins, 1979) in place of the Dirac Delta function. The operational properties of D are critical in proceeding with the formulation and will be discussed subsequently. The response of the adjoint equation, Equation (4), to application of this forcing function is denoted by a new function

$E(x,\xi)$ which replaces the free-space Green's solution $G(x,\xi)$ in Equations (1-4). In particular, the function $D(x,\xi)$ obeys the following conditions:

$$\begin{aligned}
&\text{(a)}\ \frac{\partial}{\partial x_i}[k_{ij}(x)\frac{\partial E}{\partial x_j}(x,\xi)] = -D(x,\xi) \\
&\text{(b)}\ \int_{\Omega,c} D(x,\xi)\, d\Omega(x) = 1 \qquad (5) \\
&\text{(c)}\ \int_{\Omega} f(x) D(x,\xi)\, d\Omega(x) = f(\xi) A(\xi) \\
&\text{(d)}\ A(\xi) = \int_{\Omega} D(x,\xi)\, d\Omega(x)
\end{aligned}$$

where Ω,c is a circular domain centered about the source point ξ and Ω is arbitrary in shape but encloses the source point ξ. A singular non-symmetric generalized forcing function $D(x,\xi)$ is generated by constructing this set of relations. The sampling properties of D are satisfied once the above is solved; in particular, the integral of $D(x,\xi)$ over an arbitrary domain enclosing the source point ξ, shifted at the source point, and multiplied by any arbitrary function $f(x)$, yields back the function $f(x)$ at the source point ξ multiplied by an amplification factor, $A(\xi)$. This is the direct consequence of the strongly singular and non-symmetric nature of D which is constructed based upon the spatial variation of the hydraulic conductivity.

The amplification factor $A(\xi)$ can be evaluated readily based upon the assumption that the function $E(x,\xi)$ is available. Using relation (d) in Equation (5) with the relation (a) yields the following equation for $A(\xi)$ based upon the function $E(x,\xi)$:

$$A(\xi) = -\int_{\Omega} \frac{\partial}{\partial x_i}[k_{ij}(x)\frac{\partial E}{\partial x_j}(x,\xi)]\, d\Omega(x) \qquad (6)$$

Applying the Gauss-Divergence theorem to the right hand side of the above equation provides the amplification factor as the contour integral:

$$A(\xi) = -\oint_{\Gamma} [k_{ij}(x) \frac{\partial E}{\partial x_j}(x,\xi)]\, d\Gamma(x) \qquad (7)$$

It is noted that the amplification factor explicitly depends upon the solution of the adjoint equation and the variation of the hydraulic conductivity. For the limiting case of a constant permeability, $E(x,\xi) = G(x,\xi) = -(1/2\pi k)\ln[r(x,\xi)]$, reduces to the fundamental solution of the adjoint problem: $k\nabla^2 G(x,\xi) = -\delta(x-\xi)$ where $\delta(x-\xi)$ is the shifted Dirac Delta forcing function. The amplification factor $A(\xi)$ for this case reduces to the traditional result of $k(\xi)$ for any point ξ in the interior of the domain, and to $(1/2)k(\xi)$ for any point ξ on a smooth boundary. The term $\delta(x-\xi)$ in the adjoint equation (Eq. 4) is now replaced by the generalized forcing function D. As a result, the integral equation for piezometric heads (Eq.3) becomes

$$\oint_{\Gamma} [E(x,\xi)k_{ij}(x)\frac{\partial h}{\partial x_j}(x)n_i - h(x)k_{ij}(x)\frac{\partial E}{\partial x_j}(x,\xi)n_i]d\Gamma(x) - \int_{\Omega}[h(x)D(x,\xi)]d\Omega = 0 \qquad (8)$$

Now, if the sampling property of the forcing function D is invoked, the integral equation (8) can be transformed into equation for the piezometric heads as

$$A(\xi)h(\xi) = \oint_{\Gamma} [E(x,\xi)\, k_{ij}(x) \frac{\partial h}{\partial x_j}(x)n_i - h(x)\, k_{ij}(x) \frac{\partial E}{\partial x_j}(x,\xi)n_i]\, d\Gamma(x) \qquad (9)$$

where the expression for the amplification factor is derived from Equation 7. It must be noted that the amplification factor is to be computed at all points where the head is sought, including both the surface Γ and the interior of the domain Ω. This equation can be solved by using standard BEM algorithm upon evaluating the function $E(x,\xi)$. The numerical implementation of this equation is discussed further in this paper.

GENERALIZED FUNDAMENTAL SOLUTIONS

A generalized fundamental solution for the non-homogeneous case is obtained by solving the adjoint equation of the problem, i.e. Equation 5(a). The details of the derivation of the fundamental solution for the isotropic non-homogeneous case, have been presented in Kassab and Divo (1996). In this section, the fundamental solution for the orthotropic case is detailed for a two dimensional problem. The subsequent discussion of the three dimensional case only provides the highlights of the derivation since a significant part of the development is identical to the two dimensional case.

Two Dimensional Generalized Fundamental Solution for Orthotropic Media

The fundamental solution is derived by transforming the governing equation into a polar coordinate system such that the origin of the system is located at the source point ξ as shown in Figure 1. As a result, the adjoint equation, in the local polar coordinate system, becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rk_{rr}\frac{\partial E}{\partial r})+\frac{1}{r}\frac{\partial}{\partial r}(k_{r\theta}\frac{\partial E}{\partial \theta})+\frac{1}{r}\frac{\partial}{\partial \theta}(k_{r\theta}\frac{\partial E}{\partial r})+\frac{1}{r^2}\frac{\partial}{\partial \theta}(k_{\theta\theta}\frac{\partial E}{\partial \theta})=-D(r,\theta,\xi) \qquad (10)$$

where the permeability tensor is dependent upon the spatial location of the source point and is anisotropic in nature. Thus,

$$\vec{k}(r,\theta,x)=\begin{bmatrix} k_{rr} & k_{r\theta} \\ k_{\theta r} & k_{\theta\theta} \end{bmatrix} \qquad (11)$$

The generalized singular forcing function $D(r,\theta,\xi)$ is non-symmetric around the source point and allows the selection of locally radially symmetric fundamental solution, such that $E = E(r,\xi)$. With the radial symmetry of E, the adjoint equation above reduces to the following

$$\frac{1}{r}\frac{\partial}{\partial r}(rk_{rr}\frac{\partial E}{\partial r})+\frac{1}{r}\frac{\partial}{\partial \theta}(k_{r\theta}\frac{\partial E}{\partial r})=-D(r,\theta,\xi) \qquad (12)$$

The forcing function in this equation must be non-symmetric for arbitrary variations of the k-tensor. Since the inner product of the permeability tensor and the fundamental solution E is a function of (r,θ,ξ), the right hand side function D must also be function of (r,θ,ξ) and therefore, non-symmetric in nature. Equation (12) is solved using a two-step procedure as described below. By rearranging the adjoint equation, the following convenient form may be obtained

$$\frac{\partial}{\partial r}(rk_{rr}\frac{\partial E}{\partial r})+\frac{1}{rk_{rr}}(\frac{\partial k_{r\theta}}{\partial\theta})(rk_{rr}\frac{\partial E}{\partial r})=-rD(r,\theta,\xi) \qquad (13)$$

Integrating Equation (13), using the properties of the forcing function as described by Equation (5), results in

$$\frac{\partial E}{\partial r}(r,\xi)=-\frac{f(r,\theta,\xi)}{rk_{rr}(r,\theta,\xi)} \qquad (14)$$

where the general function f is defined as

$$f(r,\theta,\xi)=e^{-\int\frac{1}{rk_{rr}}\left(\frac{\partial k_{r\theta}}{\partial\theta}\right)dr}\int rD(r,\theta,\xi)\,e^{\int\frac{1}{rk_{rr}}\left(\frac{\partial k_{r\theta}}{\partial\theta}\right)dr}dr+h(\theta) \qquad (15)$$

Since the left hand side of Equation (14) is only a function of the local radial location, the right hand side term must be such that the two functions cancel out the overall θ dependence. Therefore, a new function $g(r,\theta,\xi)$ may be defined as

$$g(r,\xi)=-\frac{f(r,\theta,\xi)}{k_{rr}(r,\theta,\xi)} \qquad (16)$$

which leads to the following relation

$$\frac{\partial E}{\partial r}(r,\xi) = -\frac{g(r,\xi)}{r} \tag{17}$$

This equation may be integrated again to yield the generalized fundamental solution, $E(r,\xi)$, as follows

$$E(r,\xi) = \int \frac{g(r,\xi)}{r} dr \tag{18}$$

To determine the value of the function $g(r,\xi)$, Equation (12) is integrated over a circular domain Ω_c centered around the source point ξ, as

$$\int_{\Omega_c} \left(\frac{1}{r}\frac{\partial}{\partial r}(rk_{rr}\frac{\partial E}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(k_{r\theta}\frac{\partial E}{\partial r}) \right) d\Omega_c = -\int_{\Omega_c} D(x,\xi)\, d\Omega_c \tag{19}$$

The right hand side integral is unity by the property of the generalized forcing function $D(x,\xi)$. The Gauss divergence theorem is applied to the left hand side of the equation to obtain

$$\oint_{\Gamma_c} \left(k_{rr}\frac{\partial E}{\partial r} n_r + k_{r\theta}\frac{\partial E}{\partial r} n_\theta \right) d\Gamma_c = -1 \tag{20}$$

where Γ_c is boundary of the circular domain, Ω_c, surrounding the source point ξ and n_r and n_θ are the radial and tangential components of the outward unit normal. On the circular boundary, $\Gamma_c = rd\theta, n_r = 1$ and $n_\theta = 0$. Thus, integrating over the circular boundary

$$\int_0^{2\pi} \left[rk_{rr}(r,\theta,\xi)\frac{\partial E}{\partial r}(r,\xi) \right] d\theta = -1 \tag{21}$$

which may be simplified, using Equation (17), as follows

$$\int_0^{2\pi} \left[g(r,\xi)\, k_{rr}(r,\theta,\xi) \right] d\theta = -1 \tag{22}$$

Since the function g is not a function of θ, it may be taken out of the integral, yielding the expression

$$g(r,\xi) = \frac{-1}{\int_0^{2\pi} k_{rr}(r,\theta,\xi)\,d\theta} \tag{23}$$

The generalized fundamental solution can now be expressed as follows

$$E(r,\xi) = -\int \frac{dr}{r\int_0^{2\pi} k_{rr}(r,\theta,\xi)\,d\theta} \tag{24}$$

This solution is valid for any arbitrary variation of hydraulic conductivity and varies with both the spatial location and orientation at any point in the domain. It may noted that for the limiting case of constant isotropic permeability k, the above equation reduces to the well-known fundamental solution to the Laplace equation in two-dimensions, $E(r) = (1/2\pi k)\ln(r)$.

The anisotropic permeability tensor in the shifted polar coordinate system can be transformed back to the corresponding tensor in Cartesian system in 2-D as follows

$$\begin{bmatrix} k_{rr} & k_{r\theta} \\ k_{r\theta} & k_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{25}$$

leading to the following relation for k_{rr},

$$k_{rr} = \cos^2\theta\, k_{xx} + \sin^2\theta\, k_{yy} \tag{26}$$

which may then be substituted into Equation (24) to obtain the following expression for the generalized fundamental solution

$$E(r,\xi) = -\int \frac{dr}{r\int_0^{2\pi}(\cos^2\theta k_{xx} + \sin^2\theta k_{yy})\,d\theta} \tag{27}$$

Thus, the variable coefficients k_{xx} and k_{yy} of the permeability tensor must be transformed into the shifted polar coordinate system before carrying out the integration. It may noted that in addition to retrieving the conventional solution for constant permeability, the above generalized fundamental solution also yields the fundamental solution for flow in an isotropic heterogeneous porous medium (Kassab and Divo, 1996).

Three Dimensional Generalized Fundamental Solution

The three dimensional fundamental solution is obtained in a manner similar to the previous case. A transformation is made to a local spherical coordinate system centered around the source point $\xi = (x_o, y_o, z_o)$. By perturbing the adjoint equation by a non-symmetric forcing function D, a generalized fundamental solution $E(r,\xi)$ may be obtained as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k_{rr}\frac{\partial E}{\partial r}\right) + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, k_{r\theta}\frac{\partial E}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(k_{r\theta}\frac{\partial E}{\partial r}\right)\right] = -D(r,\theta,\phi,\xi) \tag{28}$$

Following steps similar to the derivation in two dimensions, the generalized fundamental solution in three dimensions becomes

$$E(r,\xi) = -\int \frac{dr}{r^2\int_0^{2\pi}\int_0^{2\pi} k_{rr}(r,\theta,\phi,\xi)\,\sin\theta\, d\theta\, d\phi} \tag{29}$$

The inverse transformation from spherical to Cartesian coordinate system is given as, $x = x_o + r\sin\theta\cos\phi,\ y = y_o + r\sin\theta\sin\phi,$ and $z = z_o + r\cos\theta$. Using the orthogonal transformation as before, the expression for the k_{rr} component of the permeability tensor under orthotropic conditions, can be written in terms of the Cartesian components as,

$$k_{rr}(r,\theta,\phi,\xi) = (\sin^2\theta\cos^2\phi)k_{xx} + (\sin^2\theta\sin^2\phi)k_{yy} + (\cos^2\theta)k_{zz} \tag{30}$$

Once again, it may noted that this fundamental solution reduces to the well-known three dimensional solution for steady-state groundwater flow in a medium with isotropic constant permeability, i.e. $E = 1/(4\pi r)$.

This section was devoted to the derivation of the generalized fundamental solutions for two- and three-dimensional steady state flow in an orthotropic non-homogeneous medium. The solutions to the variable coefficient partial differential equation obtained in this process are completely general and applicable to a wide range of engineering problems governed by such equations.

NUMERICAL IMPLEMENTATION

The numerical implementation of the integral equation (Equation 9), follows standard BEM procedures for steady-state problems. The boundary of the domain is discretized using N-boundary nodes and isoparametric quadratic boundary elements are used to model the piezometric head and the fluid flux at the boundary. The algorithm developed by Kassab and Nordlund (1995) is implemented at corners at which the head is prescribed. During the collocation process, the source point is taken to each node on the boundary and the corresponding surface integrals are evaluated numerically using quadratures. The use of double precision Gauss-Kronrod ($\mathrm{G}_7 - \mathrm{K}_{15}$) based adaptive quadrature routine DQAGS from the library of the QUADPACK package (Piessens *et al*. 1980) is made for accurate integration. The boundary integral equation is written in discretized form as

$$A(\xi^p)h(\xi^p) = \sum_{q=1}^{N} G^{p,q}\left(k^q{}_{ij}\,\frac{\partial h^q}{\partial x_j}n_i\right) - \sum_{q=1}^{N} H^{p,q}h^q, \quad p = 1,2,\ldots.N \tag{31}$$

The amplification factor, $A(\xi^p)$ is evaluated numerically using the same boundary discretization as

$$A(\xi^p) = -\sum_{q=1}^{N}\oint_{\Gamma^q}\left[k^q{}_{ij}(x)\frac{\partial E}{\partial x_j}(x,\xi)\,n_i\right]d\Gamma^q(x) \tag{32}$$

where $\Gamma^q(x)$ is the qth boundary element. The application of Equation (31) to all surface nodes leads to a matrix set of equations with a standard BEM form, [H]{h}=[G]{q}. This set of equations may be solved for the unknown quantities on the boundary by introducing the boundary conditions. The integral equation (Equation 9) can also be used to solve for interior unknowns by taking the source point ξ to the interior point. It must be noted here that unlike the homogeneous case where the amplification factor for interior points $A(\xi) = 1$, the amplification factor for heterogeneous media must be computed for the interior points by using Equation (32).

There are many cases in practice in which the diagonal components of the hydraulic conductivity tensor differ substantially from each other. For instance, ratios of ten to one are often encountered in highly orthotropic media. These kind of problems pose numerical difficulties due to the high conditioning number of the coefficient matrix which amplifies errors due to boundary discretization and numerical integration. In order to handle such cases, a scaling of the axes is introduced. For a two- dimensional case, the governing equation may be written in vector form as:

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \bullet \left(\begin{bmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{bmatrix} \begin{pmatrix} \dfrac{\partial h}{\partial x} \\ \dfrac{\partial h}{\partial y} \end{pmatrix} \right) = 0 \tag{33}$$

Scaling of the x- and y-coordinates is introduced as:

$$x = \xi\sqrt{k_{xx\max}} \quad and \quad y = \eta\sqrt{k_{yy\max}} \tag{34}$$

leading to

$$\frac{\partial}{\partial\xi}\left(\frac{k_{xx}}{k_{xx\max}}\frac{\partial h}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{k_{yy}}{k_{yy\max}}\frac{\partial h}{\partial\eta}\right) = 0 \tag{35}$$

which can finally be written in transformed domain as

$$\begin{pmatrix} \dfrac{\partial}{\partial\xi} \\ \dfrac{\partial}{\partial\eta} \end{pmatrix} \bullet \left(\begin{bmatrix} k_{\xi\xi} & 0 \\ 0 & k_{\eta\eta} \end{bmatrix} \begin{pmatrix} \dfrac{\partial h}{\partial\xi} \\ \dfrac{\partial h}{\partial\eta} \end{pmatrix} \right) = 0 \tag{36}$$

Thus, the problem is solved in the transformed domain with a minimized ratio of the orthotropic conductivities. The solution in the $\xi - \eta$ domain is subsequently transformed back to the original *x*-*y* domain. The following section describes a number of numerical examples to verify the formulation presented above for the BEM solution of Darcy flow problems in orthotropic heterogeneous media.

NUMERICAL EXAMPLES

This section presents numerical examples which are used to validate the formulation described in this paper and extend the capabilities of BEM in studying more complex problems of heterogeneous orthotropic media. Quadratic isoparametric boundary elements are used for all studies and both regular and irregular boundaries are investigated. In some cases, particular spatial variations of the hydraulic conductivity are assumed to permit the derivation of analytical solutions for comparison.

Example 1: One-dimensional non-homogeneous isotropic flow

This example is selected to demonstrate that the above orthotropic formulation retrieves the correct isotropic solution. The unit square geometry and imposed boundary conditions chosen for this problem are shown in Figure 2 (a). The left and right sides of the square region are considered impermeable and thus the flow may be considered one-dimensional in nature. A unit head is specified at the bottom surface while the top surface is maintained at zero head. Figure 2 (b) shows a simple BEM model consisting of 8 quadratic elements used to solve this problem.

The permeability profile is a function of the depth (y) is taken as follows

$$k(y) = \left[e^{1.7y} - 0.46\sinh(1.7y)\right]^2 \tag{37}$$

However, this function does not lead to a closed form solution for the E function. Consequently, the above is approximated by least-squares fitting of the following polynomial

$$k_a(y) = ay^3 + by^2 + cy + d \tag{38}$$

for which the fundamental solution can be obtained analytically. In this case, *a = -1.42499; b = 0.78762; c = 1.25477;* and *d = 0.98557* . The comparison the actual and approximated permeability profiles is shown in Figure 3, and the correlation is observed to be very good at all depths. The generalized fundamental solution for this problem is then derived as

$$E(r, x_o, y_o) = \frac{1}{4\pi(ay_o^3 + by_o^2 + cy_o + d)} \ln\left[\frac{(3ay_o + b)r^2 + 2(ay_o^3 + by_o^2 + cy_o + d)}{r^2}\right] \tag{39}$$

This illustrates the procedure followed in practice when the hydraulic conductivity variation has a form which does not lead to closed-form integration of Eqs. (27) and (29) for *E*.

An exact solution to this problem is found by introducing the hydraulic conductivity in the governing equation, Eq. (1), and by subsequently seeking a particular solution to the governing PDE. This procedure is followed in all examples below for which analytical solutions are derived for comparison with BEM-computed solutions. For the above case, an analytical solution satisfying the governing Darcy equation for the hydraulic conductivity in Eq. (38) can be obtained as

$$h(y) = \frac{-\coth(1.7) + 1 - 0.46}{\coth(1.7y) + 1 - 0.46} \tag{40}$$

The comparison between the exact and BEM solution is provided in Figure 4 where the variation of the hydraulic head with depth is plotted. It is evident from the Figure that, even with a relatively coarse 8 quadratic element (15 node) model, the two solution agree well at all depths. This verifies that the above formulation reduces to the correct solution in the limit of non-homogeneous isotropic media. Attention is now given to a series of two- and three- dimensional non-homogeneous orthotropic problems.

Example 2: Two-dimensional homogeneous orthotropic flow beneath a dam

The next example is a two-dimensional application of seepage through a homogenous orthotropic soil beneath an impervious dam. This example is selected to demonstrate that the above formulation retrieves the correct homogeneous orthotropic solution The geometry and prescribed boundary conditions are shown in Figure 5. The width of the bottom of the dam and the thickness of the soil layer are both taken as unity for convenience. The domain on both sides of the dam was truncated at a distance equal to 5 times the dam width. The homogeneous orthotropic variation of hydraulic conductivity is assumed to be

$$k_{ij} = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

An exact solution for the piezometric head anywhere in the domain may be evaluated as

$$h(x, y) = \frac{-x^2 + 10y^2 - 6xy + 20x + 25y}{10} + 100 \tag{42}$$

In this example and all subsequent 2-D seepage examples (3 and 4), forty quadratic boundary elements are used. Figures 6(a-b) compare the BEM solution with the analytical solution for the distribution piezometric heads within the soil domain. It is evident from the contours that the two results are very close. The relative percent error in the heads computed by BEM may be defined as ($|H_{exact} - H_{BEM}| / H_{exact}$ x100) and is plotted in Figure 6(c). This plot indicates that the maximum error is around 0.092 %, verifying that the correct homogeneous orthotropic solution is obtained. Having demonstrated both limiting cases of the above formulation, attention is now given to flows in heterogenous media.

Example 3: Two-dimensional orthotropic heterogeneous flow in a square region

Building upon the previous groundwater flow beneath an impervious dam, the following heterogeneous variation of the hydraulic conductivity is considered in this example,

$$k_{ij}(x, y) = \begin{bmatrix} 8x + 25y + 270 & 0 \\ 0 & x + 2y + 30 \end{bmatrix} \tag{43}$$

For this case, the exact solution for piezometric heads at any point in the body can be evaluated as

$$h(x, y) = \frac{-4y^2 + 24x + 4xy + 24y}{30} + 100 \tag{44}$$

The geometry and boundary condition descriptions are the same as shown in Figure 7 and once again, forty quadratic elements are used for the solution.

Figures 8(a-b) display the comparison of the results obtained by BEM analysis and the analytical solution derived in Equation (44). The correlation between the two sets of results is excellent with the maximum relative error of the order of 0.213 % as shown in Figure 8[c]. This problem validates the generalized BEM formulation for an orthotropic heterogeneous soil medium.

Example 4: Orthotropic heterogeneous flow beneath a dam with prescribed heads

Having established the accuracy of the BEM formulation for heterogeneous flow, we now consider the seepage below the dam with specified upstream and downstream conditions. The problem description and boundary conditions are shown in Figure 9. Three cases of permeability variations in the soil are considered, namely,

(i) Homogeneous orthotropic

$$k_{const} = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \qquad (45)$$

(ii) Heterogeneous orthotropic with a linear variation

$$k_{linear} = \begin{bmatrix} 10(x+12y+40) & 0 \\ 0 & x+12y+40 \end{bmatrix} \qquad (46)$$

(iii) Heterogeneous orthotropic with a quadratic variation

$$k_{quad} = \begin{bmatrix} 10(x^2+10xy+45y^2+5x+40y+300) & 0 \\ 0 & (x^2+10xy+45y^2+5x+40y+300) \end{bmatrix}$$

(47)

Note that in all cases the ratio of hydraulic conductivity in the *x*-direction to the *y*-direction is ten to one. The BEM solutions for the head in the soil domain for all three cases are shown in Figures 10 (a-c). The contours of piezometric heads represent equipotential lines. This demonstrates a practical application of our formulation to a problem in which heads are prescribed upstream and downstream of a dam. However, no analytical solutions are available for comparison. In the next example, our BEM predicted solution is compared to results of previous studies obtained by a dual reciprocity BEM and by FEM.

Example 5: Yun-Lin Aquifer Problem (Lafe and Cheng, 1987)

In this example, we consider the Yun-Lin aquifer on the island of Taiwan studied by Lafe and Cheng (1987). The simulated aquifer geometry and values of the transmissivity at selected interior locations is shown in Figure 11. The aquifer transmissivity plays the same role as the hydraulic conductivity, and the governing equation for the groundwater flow remains Equation (1). Based upon the discrete values provided in Lafe and Cheng (1987), the conductivity may be approximated by least-squares fit of the following polynomial:

$$k(x,y) = C_1 + C_2 xy + C_3 y^2 + C_4 x + C_5 y + C_6 \tag{48a}$$

for which the following constants are obtained:

$$\begin{aligned}&C_1 = 0.54828;\ C_2 = 0.10968;\ C_3 = 0.33679;\ C_4 = -18.65872;\ C_5 = -21.34066;\\ &C_6 = -48.77641\end{aligned} \tag{48b}$$

Figure 12 shows a contour plot of the approximated conductivity variation within the aquifer domain. The aquifer is bounded on two sides by impermeable formations and consists of a single layer. The aquifer is modeled using 44 equally-spaced quadratic boundary elements, and it may noted that in contrast to previous FEM and BEM studies, no interior elements are required. The generalized fundamental solution for this case is found to be

$$E(r, x_o, y_o) = \frac{\ln\left[(2K_1 + (C_1 + C_3)r^2\right]^{1/2} - \ln(r)}{2\pi K_1} \quad (49a)$$

where

$$K_1 = C_1 + C_2 x_o y_o + C_3 y_o^{\ 2} + C_4 x_o + C_5 y_o + C_6 \quad (49b)$$

The distribution of the head around the boundary of the aquifer starting at $x = 4$, $y = 0$ and going counterclockwise, is shown in Figure 13. The next figure (Figure 14) shows the contours of piezometric heads within the domain of the aquifer. Based upon this figure, Table 1 presents the comparison of the heads obtained from the present BEM formulation with the results of Lafe and Cheng (1987) and the FEM results presented in Lafe *et al*. (1981) at selected interior points. In general, good agreement is observed throughout most of the domain. However, the present BEM solution approximates the transmissivity distribution in a general manner and does not restrict the principal axes of the transmissivity tensor to be parallel to the global axes directions as is done in the previous works. This may explain the differences in the solution, particularly in the zones where there are significant changes in the transmissivity values over short distances (i.e. the left side of the aquifer). Hence, the present solution may be considered to a more accurate representation of the general variation of the piezometric head in the aquifer.

Table 1 Comparison of the Piezometric Heads at Selected Interior Points

x	*y*	Head (m)		
(km)	*(km)*	BEM	Lafe and Cheng	FEM
8	4	2.621	2.43	3.9
8	8	3.25	4.72	3.7
8	16	5.064	9.39	n/a
12	8	4.968	9.93	9.1
12	24	11.943	18.68	8.6
16	12	11.621	21.42	19.8
20	16	30.852	32.4	33.1
24	24	43.888	43.07	41.4
24	32	43.847	45.04	41.6
32	16	55.072	51.08	54.7
36	24	56.336	54.61	58.8

36	32	54.939	53.46	58.9
48	28	57.345	57.01	58.3

Example 6: Three-dimensional Orthotropic Heterogeneous Field Problem

This example illustrates the application of the proposed BEM formulation to a three-dimensional problem. A simulated field problem of seepage beneath a dam that is inclined to the direction of flow is presented. Figure 14 describes the geometry and boundary conditions specified for this case. The orthotropic non-homogeneous variation of the hydraulic conductivity is assumed to be

$$k(x,y,z)=\begin{bmatrix}10(x+y+6z+45)^2 & 0 & 0\\ 0 & 10(x+y+6z+45)^2 & 0\\ 0 & 0 & (x+y+6z+45)^2\end{bmatrix} \tag{50}$$

The pre-imposed piezometric head on the top surface is given by

$$h(x,y,z)=\frac{19x+14y+40z+3xy+25xz+11yz}{x+y+6z+45}+100 \tag{51}$$

The BEM solution for the piezometric heads on the outer surface of the soil domain is presented in Figure 15(a) while the exact solution is shown in Figure 15(b). The two solutions are observed to be very close and the maximum relative error is found to be about 0.4%. The distribution of this error is plotted on the surface in Figure 15(c) and verifies the three-dimensional fundamental solution in Eq. (29).

Example 7: Three-dimensional Orthotropic Heterogeneous Field Problem

In the final example, the field problem of seepage beneath a dam inclined to the direction of flow is considered again, however the head is prescribed upstream and downstream of the dam in analogy to the 2-D example (4). Figure 16 illustrates the geometry and boundary conditions specified for this case. The orthotropic non-homogeneous variation of the hydraulic conductivity is taken as in Eq. (51).

The BEM solutions for the piezometric heads on the outer surface of the soil domain is presented in Figure 17(a). The corresponding solutions on the mid-planes and the three inside faces of the aquifer are shown in Figures 17(b-c). This concludes successful verification of the proposed generalized BEM formulation for Darcy flow in non-homogenous orthotropic flows.

CONCLUSIONS

This paper presents a BEM solution for Darcy groundwater flow through orthotropic non-homogenous media. A generalized fundamental solution is derived based upon a singular, non-symmetric generalized forcing function. The fundamental solutions derived may be represented as the local radially symmetric response to this forcing function at the source point which vary within the domain based on the location of the source point. The forcing function and the generalized fundamental solution are both functions of the hydraulic conductivity of the medium. An appropriate boundary integral equation is derived from the governing equations and then numerically implemented to facilitate the solution of realistic problems.

Several numerical examples considering the limiting cases of homogeneous and non-homogeneous orthotropy are solved for both two- and three-dimensional problems. Excellent correlation is obtained with the current formulation. Problems of regular and irregular boundaries are considered and provide confidence in the use of this BEM formulation as a powerful numerical tool.

APPENDIX I. REFERENCES

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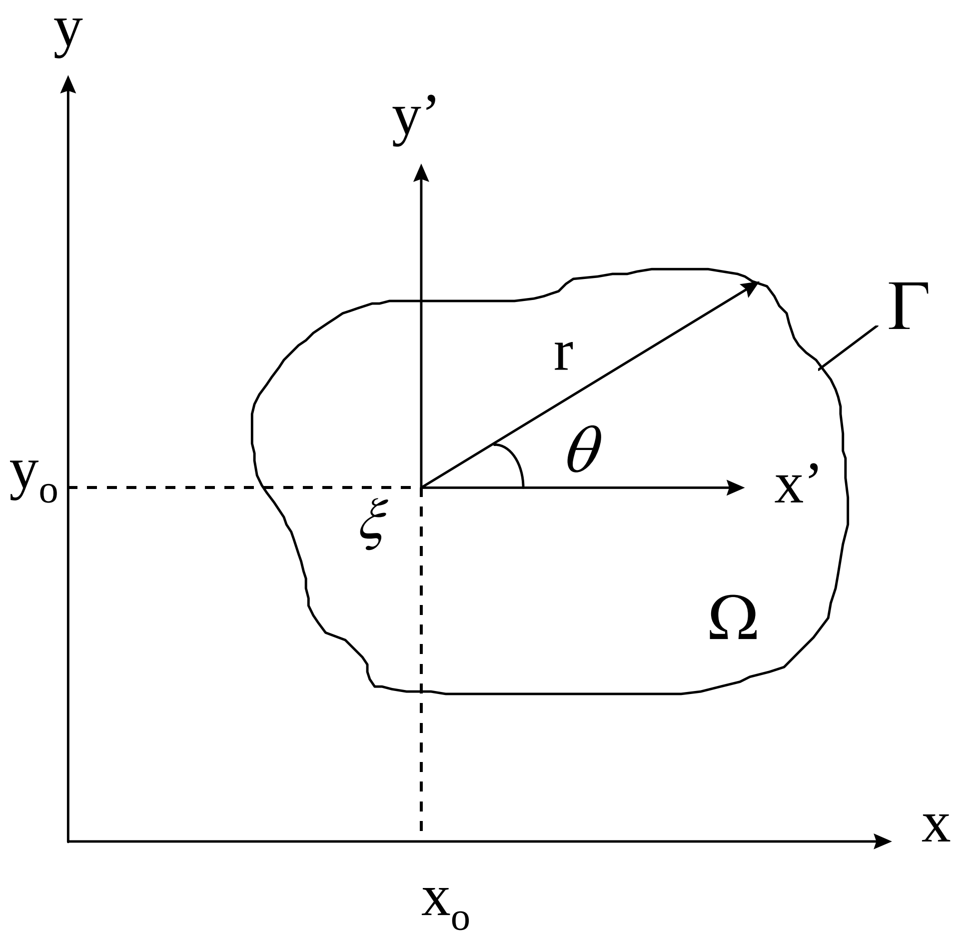

Figure 1. Local $r-\theta$ polar coordinate system centered at (x_o, y_o)

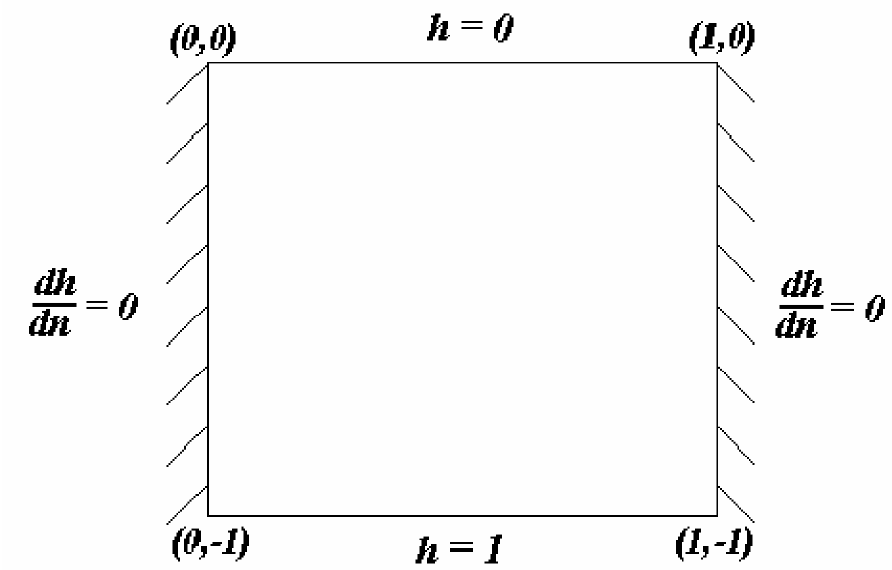

Figure 2 (a). Description and boundary conditions for one-dimensional flow problem.

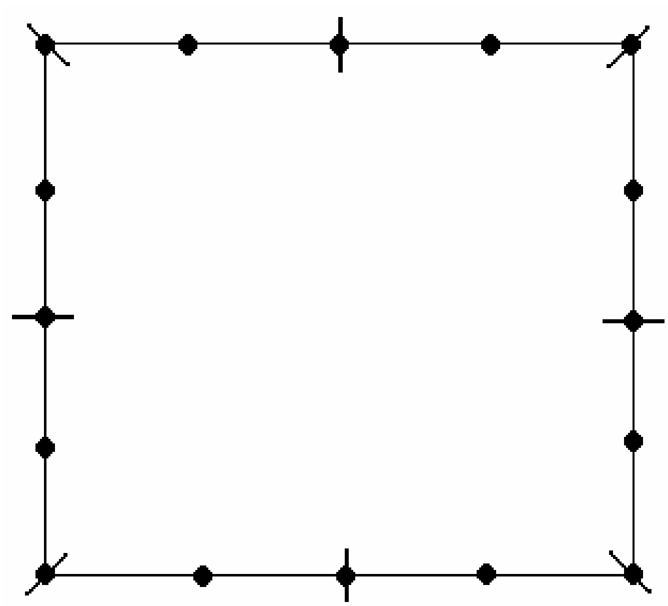

Figure 2 (b). BEM model for one-dimensional flow problem.

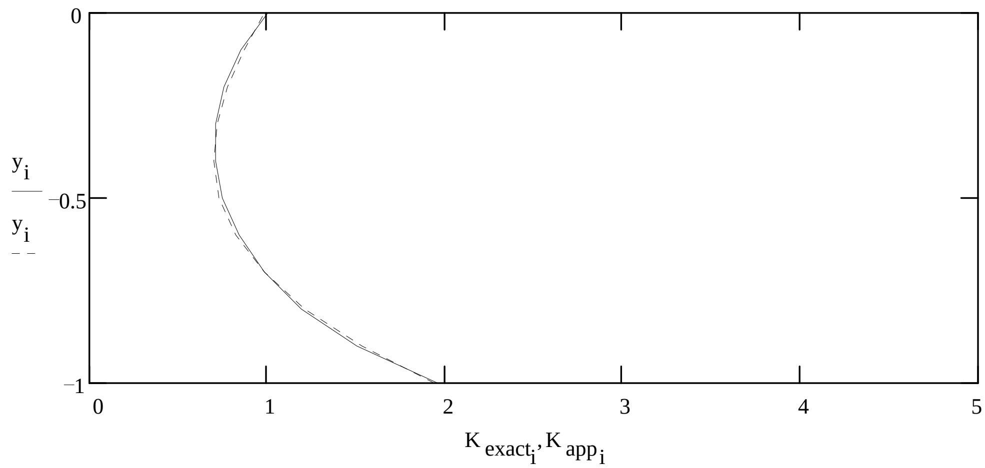

Figure 3. Comparison of the exact and approximated permeability profiles.

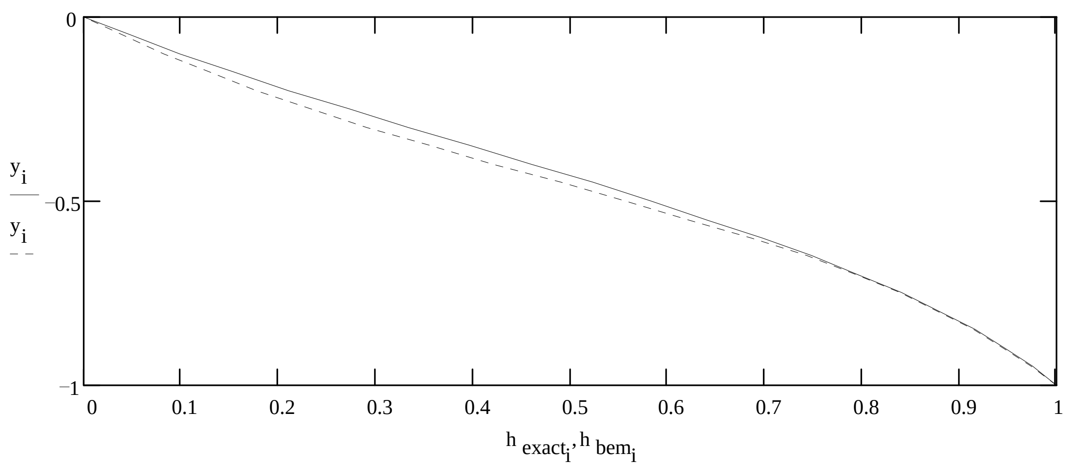

Figure 4. Comparison of the exact and BEM head distribution with depth.

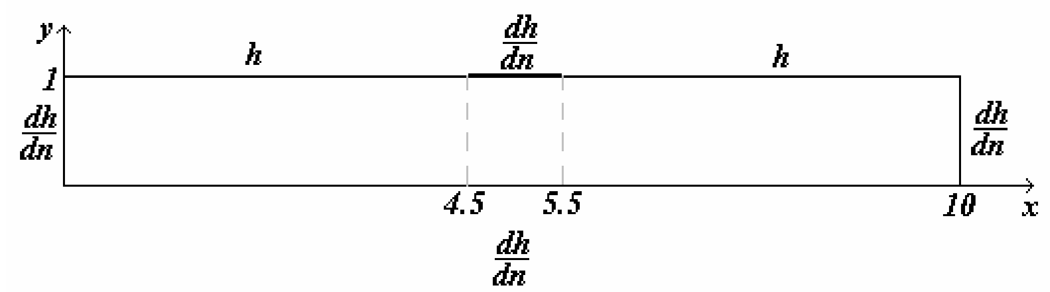

Figure 5. Description of the geometry and boundary conditions.

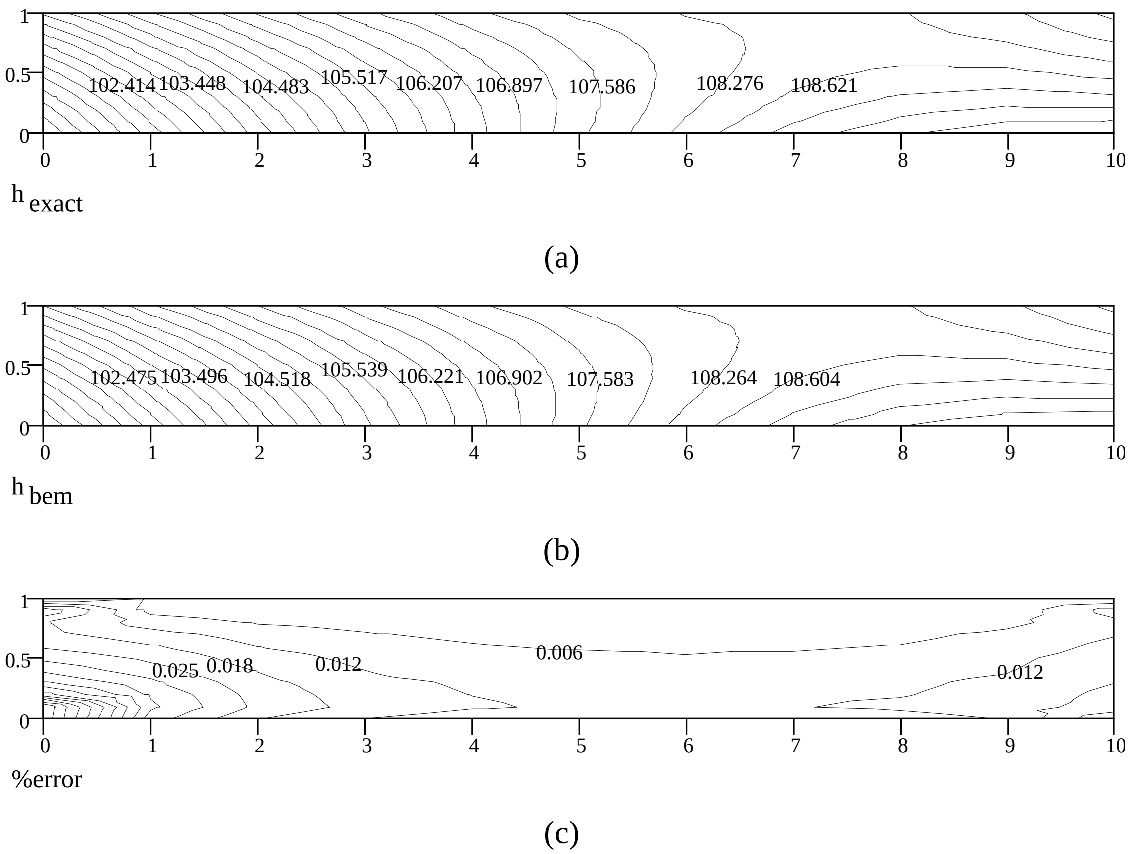

Figure 6. Contours of isoheads (a) exact solution; (b) BEM solution; (c) relative percentage error.

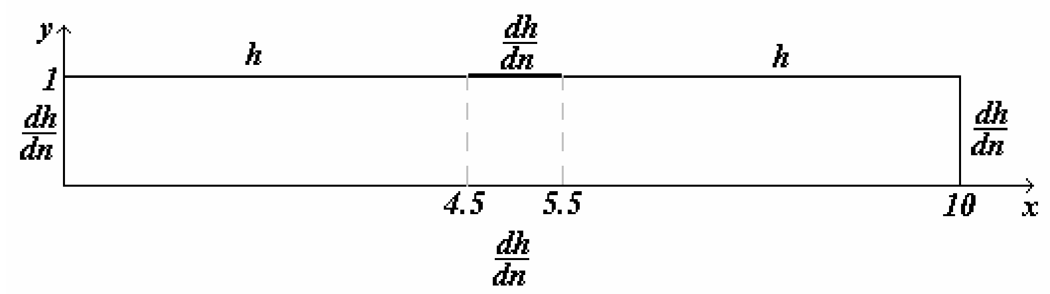

Figure 7. Description of the geometry and boundary conditions.

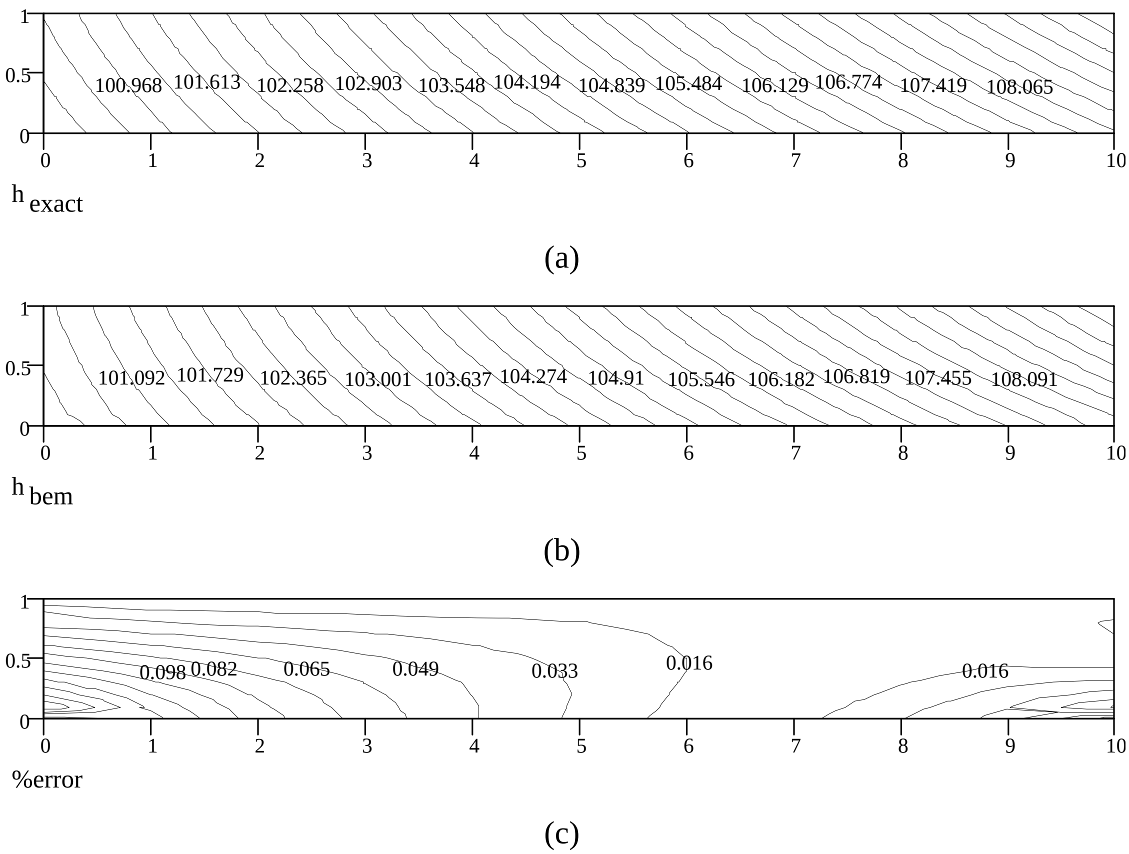

Figure 8. Contours of isoheads (a) exact solution; (b) BEM solution; (c) relative percentage error.

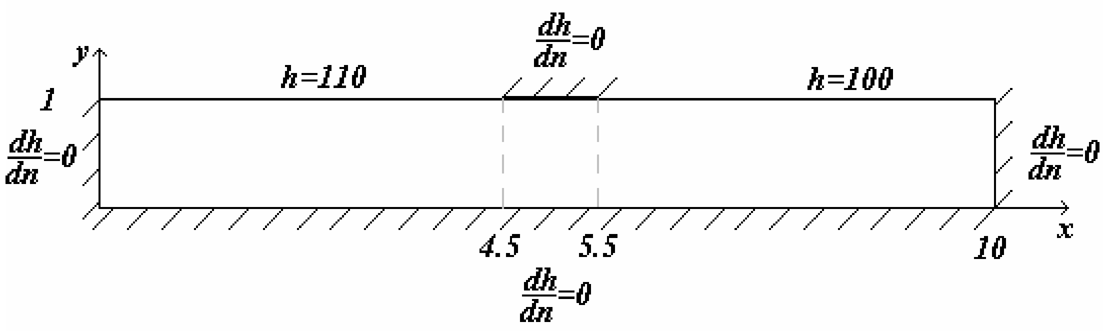

Figure 9. Description of the geometry and boundary conditions.

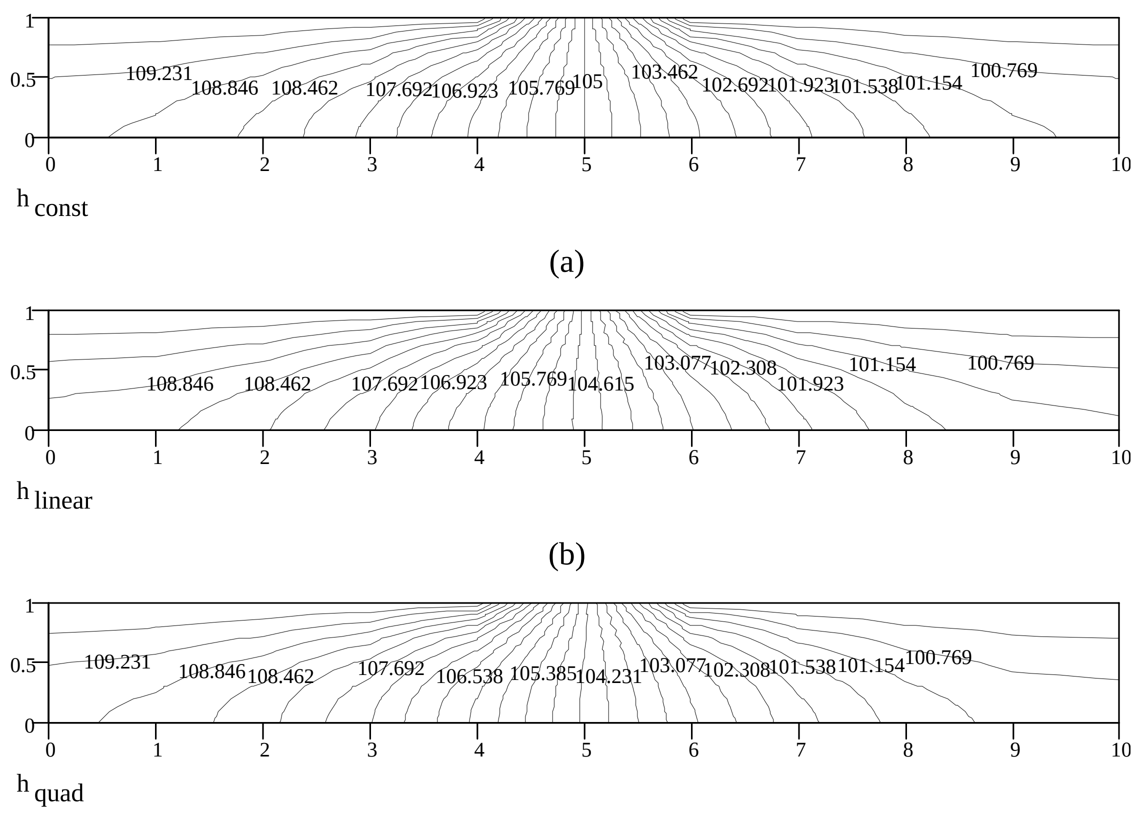

Figure 10. BEM solutions for piezometric heads (a) constant variation; (b) linear variation; (c) quadratic variation.

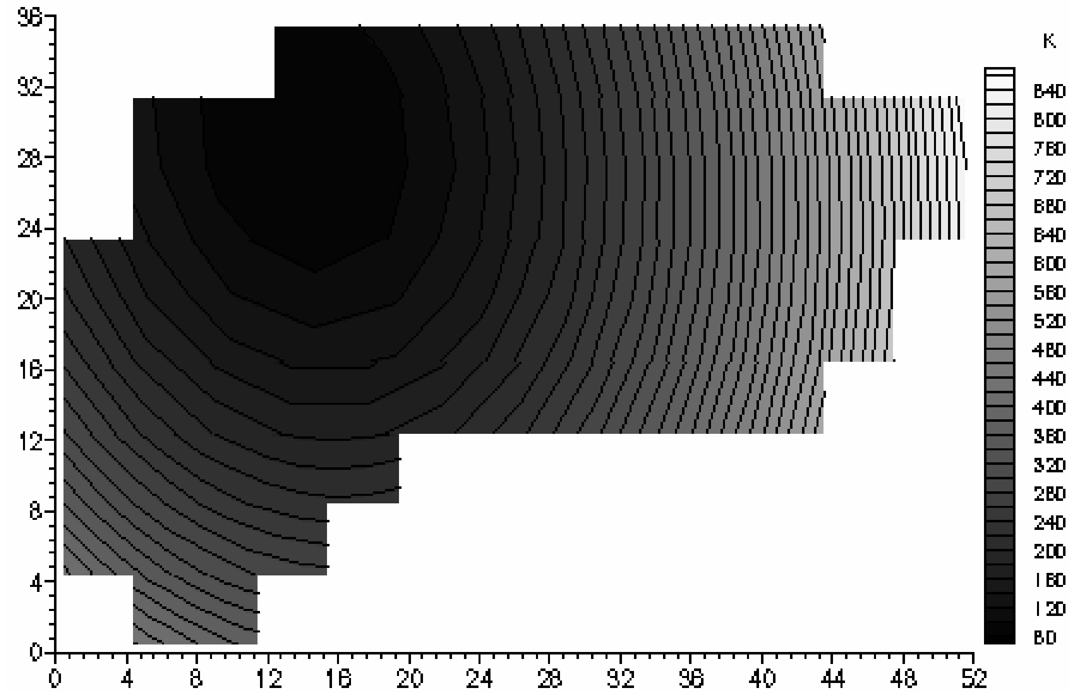

Figure 11. Contour plot of the approximate hydraulic conductivity in the geometry of the aquifer.

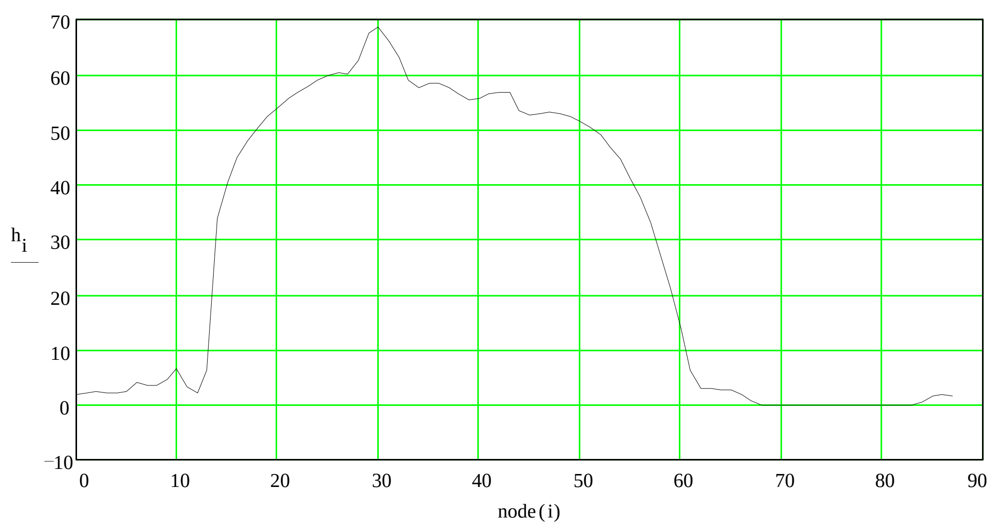

Figure 12. Piezometric head distribution around the boundary from node 1 to node 88 starting at and going counterclockwise.

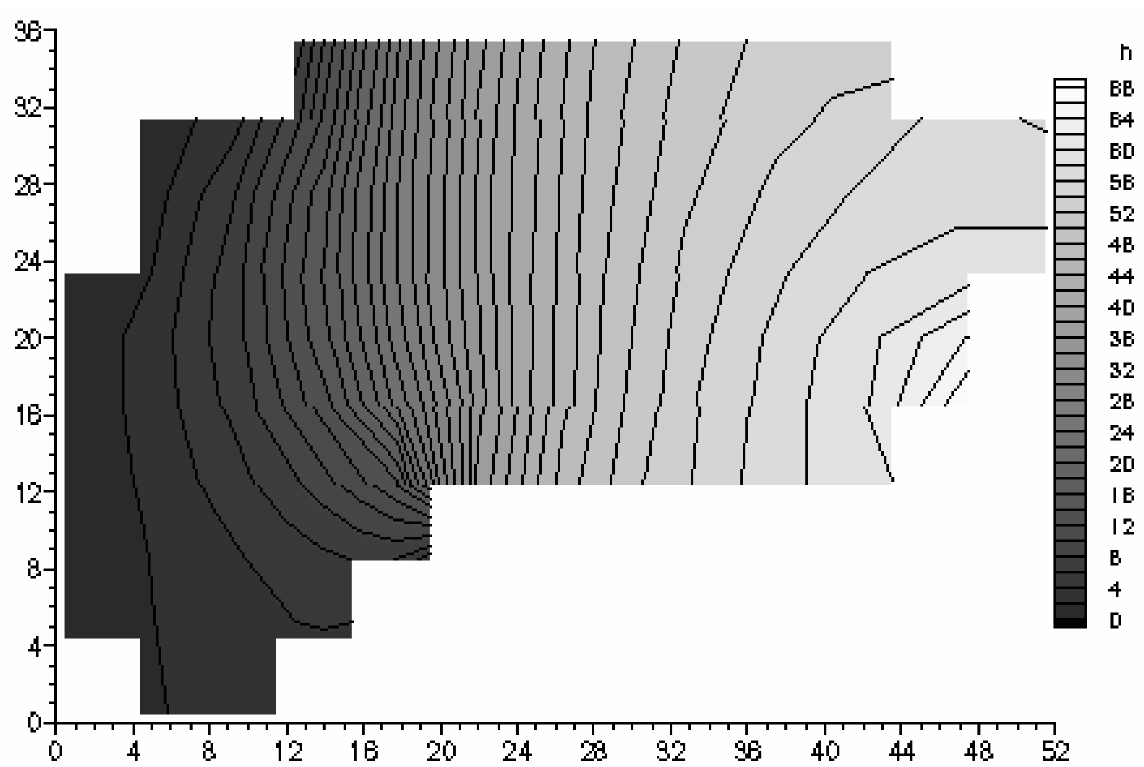

Figure 13. Contour plot of the piezometric head distribution in the aquifer.

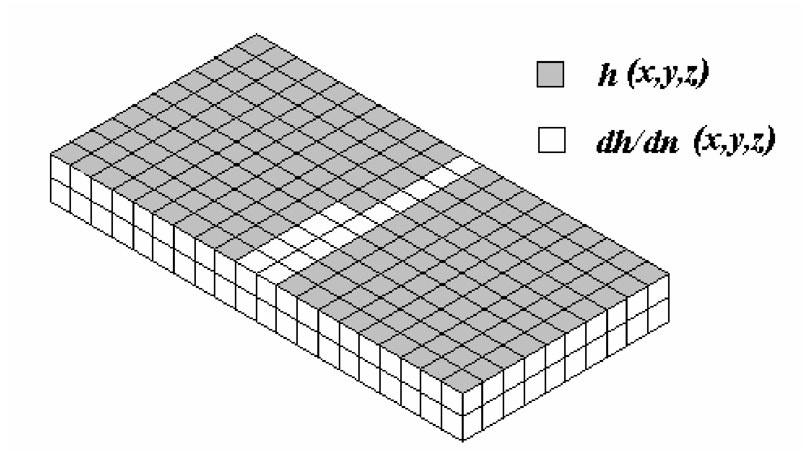

Figure 14. Description of the geometry, discretization and boundary condition.

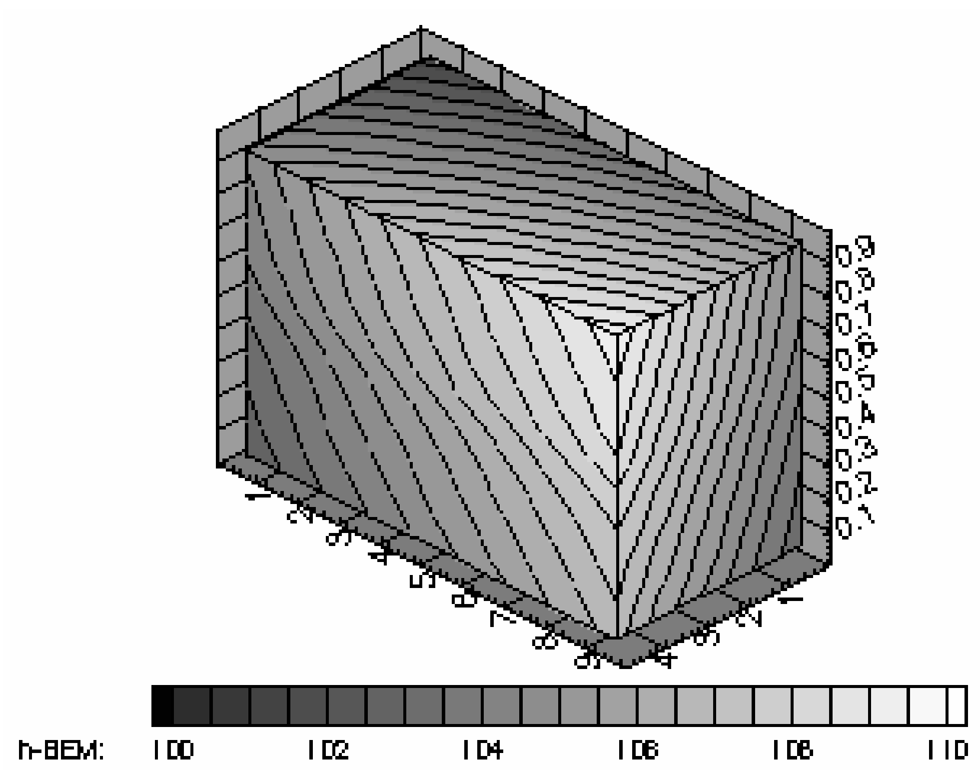

Figure 15(a). BEM solution for the three-dimensional orthotropic heterogeneous problem with pre-imposed heads on the top surface.

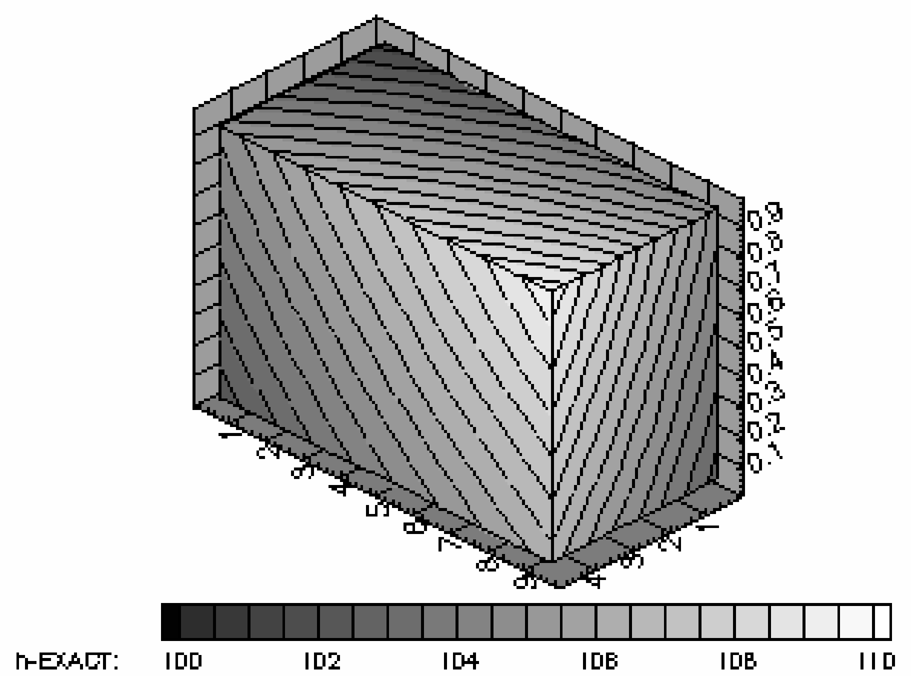

Figure 15(b). Exact solution for the three-dimensional orthotropic heterogeneous problem with pre-imposed heads on the top surface.

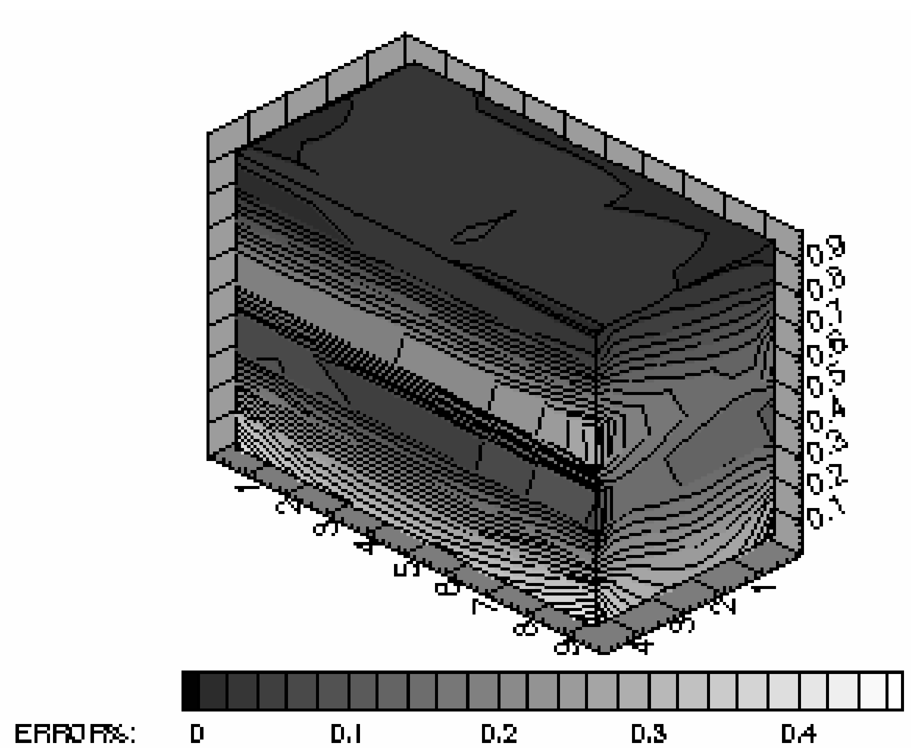

Figure 15(c). Relative percentage error between the exact solution and the BEM solution.

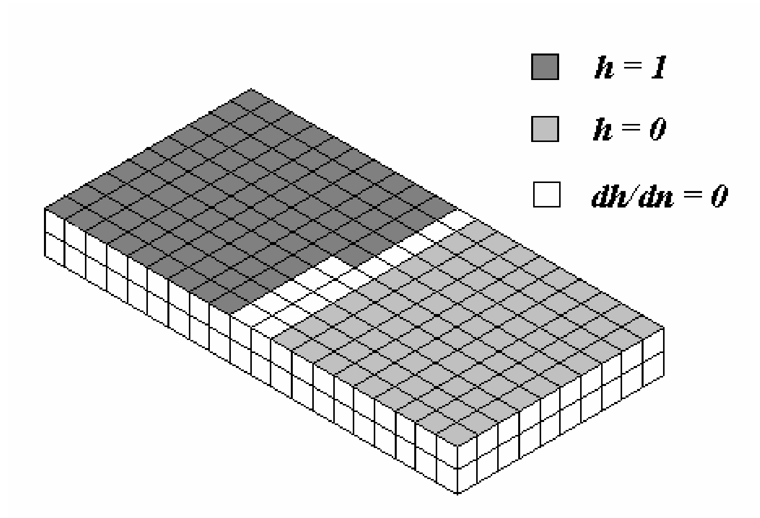

Figure 16. Description of the geometry, discretization and boundary condition.

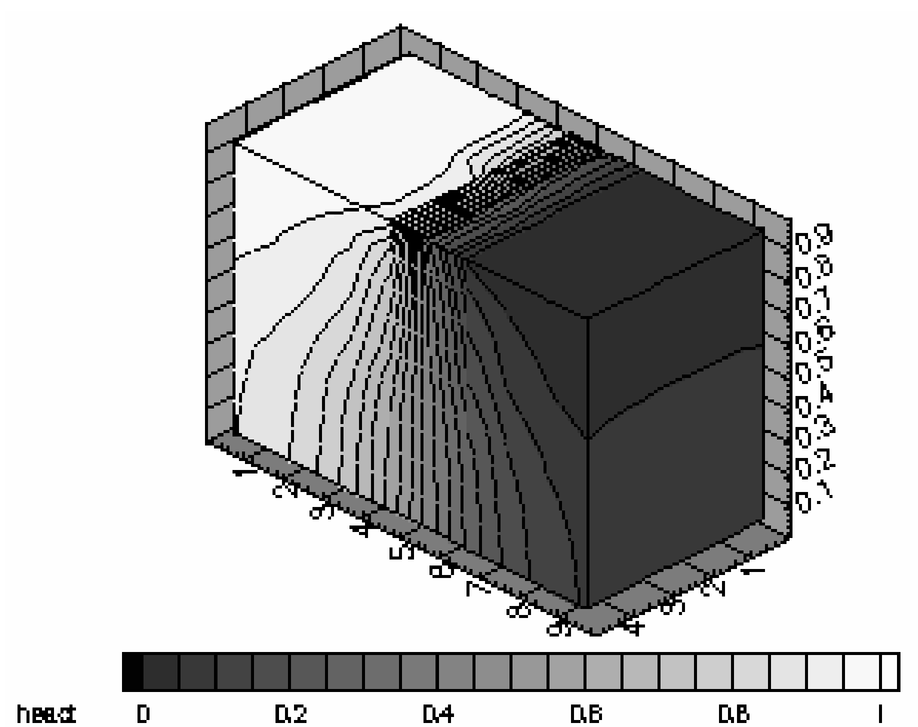

Figure 17(a). BEM solution for piezometric heads on the outside planes.

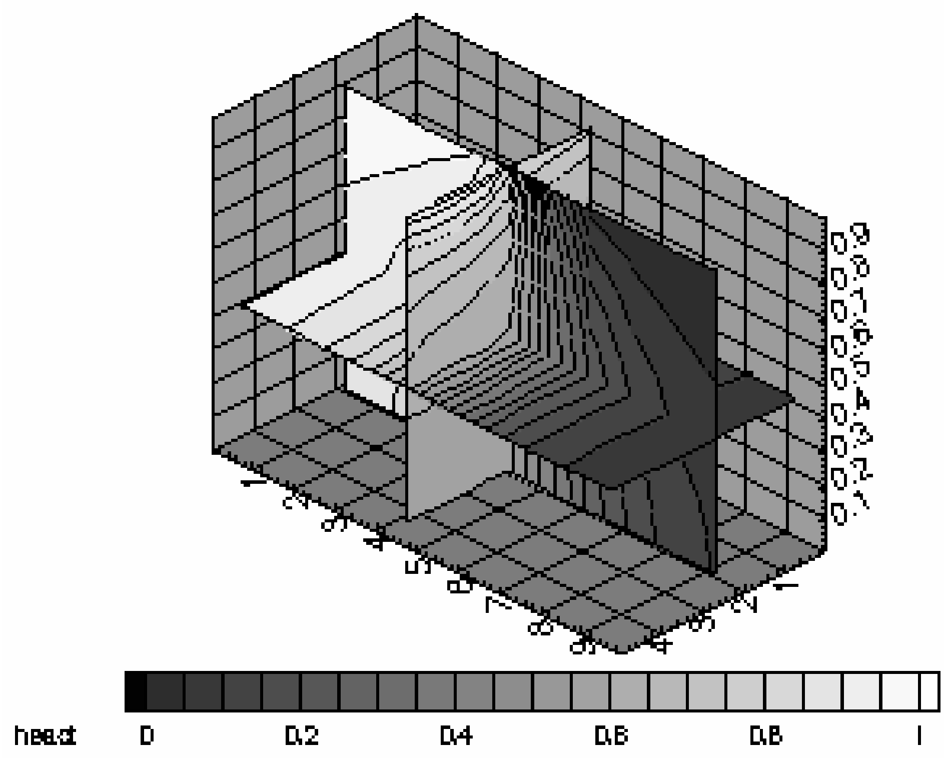

Figure 17(b). BEM solution for piezometric heads on mid-planes.

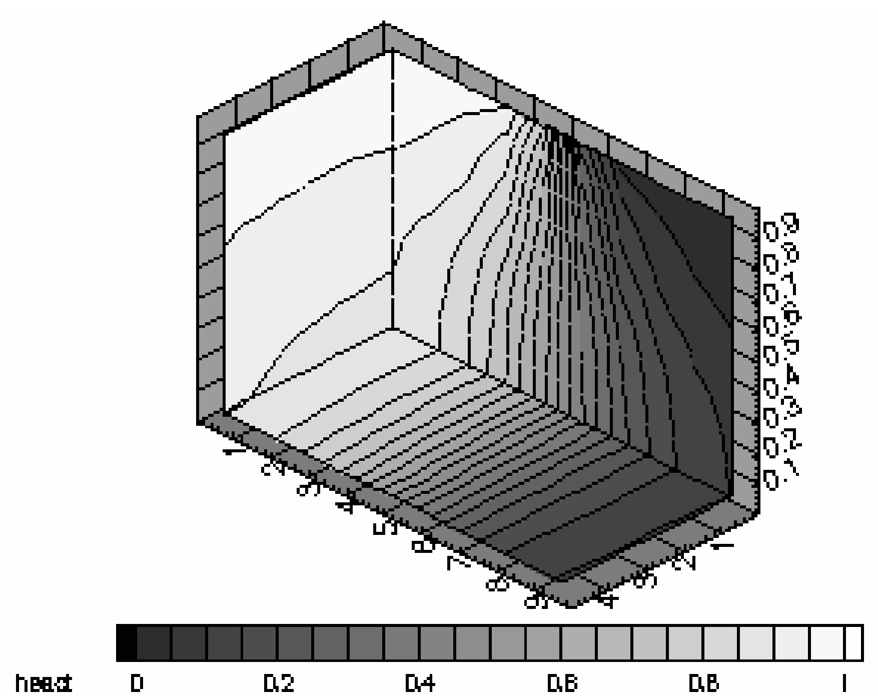

Figure 17(c). BEM solution for piezometric heads on the inside planes.